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A Mathematical Analysis of a Game of Craps

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city in China. This is her third year in the U.S. This research began in the fall of 2013 as part of an honors contract in Probability Theory (MATH 403) directed by Dr. Hang-Ling Chang, Professor of Mathematics. This research experience not only strengthened her skills in logical thinking and research, but also helped her to decide her professional future. Yaqin will pursue a PhD in the field of Probability and Statistics. She is very appreciative of the mentorship and encouragement given by Dr. Chang.

The game of craps is an extremely popular game offered by casino operators. There are some 40 different types of bets that one can place each time the game is played. One of the best bets from a player's point of view is the Pass Line bet. The probability of winning a Pass Line bet is almost the same as the probability of losing ($\frac{244}{495}$ versus $\frac{251}{495}$) as we will derive rigorously in this article. Since the "house" has such a small advantage over the players, many players possess the illusion that they have pretty good chance of making a profit or even making a living by playing this game. This article will show that this really is not the case. We will show that the probability of making a profit is almost zero if one plays the game regularly for a relatively long period of time.

I. Introduction

Craps is a dice based game where multiple players make bets against the "house" by placing chips on the appropriate part of the layout of a special table before each roll. The layout is a table cloth made of felt that displays the various betting possibilities. According to Botermans & Fankbonner (2008), "Craps is derived from an English dice game Hazard. In North America, Hazard began to spread, starting in 1800 among the black residents of New Orleans" (p. 541-42). People who play craps often believe that they have opportunities to win large amounts of money and pull down the house -- as long as they do not bet too much money all at once, and keep playing. In this article, we will show mathematically that this is not the case. There are about 40 different types of bets that can be made on a craps layout, but most of them, like the Proposition and the Hardway Bets, have terrible odds that we should avoid (Ortiz, 1986).

According to Derousseau (2007), the game of craps, as a whole, arguably gives the best odds of winning among all casino games. As a matter of fact, as we will show in this article, among all the different bets one can place in the game of craps, the Pass Line bet regards one of the best bets due to its high winning probability. In this article, we will concentrate on deriving results for a Pass Line bet. Unlike most of the literature that uses a heuristic method, we will derive all our results, including the probability of winning by using a completely rigorous mathematical approach. Ultimately, we will show that the probability of coming out ahead (making a profit) is actually slim to none if you keep playing- as problem gamblers do, who comprise 1.1 percent of the adult population of the United States and Canada, (Shaffer, Hall & Vander, 1999).

II. Description of the Pass Line Bet

We know that the game of Craps is played by rolling two dice on the Crap table. Players take turns to be the thrower. Players should make their bets before the thrower begins his/her roll, this is known as the come out roll. A thrower will continue to be the thrower until he/she “sevens out” (explained later). When that happens, another player will become the new thrower. Based on the rules described by Morehead, Frey, and Mott-Smith (1991), one of the following three things will happen on the come out roll:

- (1) A sum of 7 or 11 is rolled. The game ends. The same thrower will then make another come out roll after all players placed their new bets.
- (2) A sum of 2, 3, or 12 is rolled. The game ends as well. All Pass Line bets are lost. The same thrower will then make another come out roll after all players placed their new bets.
- (3) A sum of 4, 5, 6, 8, 9, or 10 is rolled and that sum becomes “the point”. The thrower will continue to roll the dice until either the point is rolled again or a sum of 7 is rolled. If the thrower makes the point (rolled the “point” again before a sum of 7) or “sevens out” (rolled a sum of 7 before making the point), the game will end. If the thrower makes the point, all Pass Line bets win and the same thrower will remain the thrower for the next game. If the thrower sevens out, all Pass Line bets are lost and the dice will be passed on to a new thrower.

III. Winning Probability and Expected Profit of a Pass Line Bet

A. Payoff Schedule: Before analyzing the winning probability and expected profit of a Pass Line Bet, it is necessary to know the payoff schedule. Suppose that a player makes a \$90 Pass Line bet, one of the following three cases will happen:

Case 1: If the come out roll yields a sum of 7 or 11, then the player makes a \$90 profit.

Case 2: If the come out roll yields a sum of 2, 3 or 12, then the player loses his /her \$90 bet.

Case 3: If a point (a sum of 4, 5, 6, 8, 9, or 10) is rolled on the come out roll, the thrower will continue to roll the dice until either the point is rolled again or a sum of 7 is rolled. If the point occurs first, then the player wins \$90. If the sum of 7 occurs first, then the player loses and the house wins \$90.

B. Winning Probability: In this section, I will compute the winning probability and net profit of a Pass Line bet. To start, I am going to explain two concepts in probability theory: mutually exclusive events and independent events.

Mutually Exclusive Event: According to Hsu (1996), events $E_1, E_2, E_3, \dots, E_n$ are said to be mutually exclusive if no two events can occur at the same time. The probability that at least one of the mutually exclusive events will occur is

$$P(E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n) = P(E_1) + P(E_2) + P(E_3) + \dots + P(E_n)$$

Independence: According to Hsu (1996), events $E_1, E_2, E_3, \dots, E_n$ are said to be mutually independent if the occurrence of one event does not affect the probability of the others. For example, when the thrower throws two dice, the probability of throwing a sum of k on any roll is not affected by previous outcomes. The probability of successive events is

$$P(E_1 \cap E_2 \cap E_3 \cap \dots \cap E_n) = P(E_1) * P(E_2) * P(E_3) * \dots * P(E_n)$$

On each roll of the game, the thrower throws two six-sided standard dice at the same time. There are 36 possible outcomes $(1,1), (1,2), (1,3), \dots, (6,5),$ and $(6,6)$, where (i,j) stands for the outcomes when the face value on die 1 is i and the face value on die 2 is j ($(i,j) = 1,2,3,4,5,6$).

The sum of the two face values can be any integers from 2 to 12. There are 6 ways to roll a sum of 7 $[(1,6), (6,1), (3,4), (4,3), (2,5), (5,2)]$ and 2 ways to roll a sum of 11 $[(5,6), (6,5)]$. Therefore, the probability of rolling a sum of 7 is 6 out of 36, the probability of rolling a sum of 11 is 2 out of 36. To compute the winning probability, we define the event R_k as event that a sum of k ($k = 2, 3, 4, \dots, 12$) is rolled on a single roll of the dice. The probability of rolling a sum of k is shown in Table 1 above, right:

According to the rules of the game described in section II, there are two ways to win the game; roll a sum of 7 or 11 on the come out roll or establish a point on the come out roll, then make the point afterwards. We will use W_F to denote the event that the player wins on the 1st roll of the dice. According to Table 1, the probability of rolling a sum of 7 is $\frac{6}{36}$, and the probability of rolling a sum of 11 is $\frac{2}{36}$. Since the events R_7 and R_{11} are mutually exclusive, therefore, the probability of winning on the first roll

$$\text{Is } P(W_F) = P(R_7) + P(R_{11}) = \frac{8}{36} .$$

Table 1. Probability of Rolling a Sum of k

k	P(R _k)
2	$\frac{1}{36}$
3	$\frac{2}{36}$
4	$\frac{3}{36}$
5	$\frac{4}{36}$
6	$\frac{5}{36}$
7	$\frac{6}{36}$
8	$\frac{5}{36}$
9	$\frac{4}{36}$
10	$\frac{3}{36}$
11	$\frac{2}{36}$
12	$\frac{1}{36}$

Since the events $W_{i,2}, W_{i,3}, \dots$ are mutually exclusive, therefore,

$$W_i = \bigcup_{j=2}^{\infty} W_{i,j}$$

and

$$P(W_i) = \sum_{j=2}^{\infty} P(W_{i,j}) = P(W_{i,2}) + P(W_{i,3}) + \dots + P(W_{i,j}) + \dots = P(R_i) * P(R_j) + P(R_i) * [1 - P(R_i) - P(R_j)] * P(R_j) + \dots + P(R_i) * [1 - P(R_i) - P(R_j)]^{j-2} * P(R_j) = P(R_i)^2 * \sum_{k=0}^{\infty} [1 - P(R_i) - P(R_j)]^k \quad (1)$$

Note that formula (1) is a geometric series with a common ratio of $r = [1 - P(R_i) - P(R_j)]$ and the first term $a = P(R_i)^2$. We know that the sum of the sum of a infinite geometric series with a common ratio r and the first term is a is $s = \frac{a}{1 - r}$.

Therefore, $P(W_i) = \frac{P(R_i)^2}{1 - [1 - P(R_i) - P(R_j)]}$.

Hence, $P(W_i)$ can be simplified to $\frac{P(R_i)^2}{P(R_i) + P(R_j)}$.

Table 2 summarizes the values of $P(W_i)$ for all $i = 4, 5, 6, 8, 9,$ and 10 .

We will use $W_i (i=4, 5, 6, 8, 9, \text{ or } 10)$ to denote the event that player wins by making the point i . We further define $W_{i,j}$ as the event that the player wins on the point i on the j^{th} ($j=2,3,4,\dots$) roll. In order to win a game on a point of i , one of the followings must happen:

- (1) The player wins on the second roll: the sum of the 2nd roll must be i , therefore the probability of winning on the second roll is $P(W_{i,2}) = P(R_i) * P(R_i)$ by independence.
- (2) The player wins on the third roll: the sum of the second roll must be neither the sum of point i nor 7. Therefore, winning by making the point i on the 3rd roll is $P(W_{i,3}) = P(R_i) * [1 - P(R_i) - P(R_7)] * P(R_i)$, where $[1 - P(R_i) - P(R_7)]$ is the probability of rolling a sum that is neither the sum of i nor 7.

To summarize, the probability of making the point i on the j^{th} is: $P(W_i) = P(R_i) * [1 - P(R_i) - P(R_7)]^{j-2} * P(R_i)$, where $j = 2, 3, 4, \dots$

We will use W_p to denote the event that the player wins on point 4, 5, 6, 8, 9, or 10. Since the events $W_4, W_5, W_6, W_8, W_9,$ and W_{10} are mutually exclusive, the probability of winning with a point is:

$$P(W_p) = P(W_4) + P(W_5) + P(W_6) + P(W_8) + P(W_9) + P(W_{10}) = \frac{1}{36} + \frac{2}{45} + \frac{25}{396} + \frac{25}{396} + \frac{2}{45} + \frac{1}{36} = \frac{134}{495}$$

Let W denote the event of winning on a Pass Line bet. Since the event W_p and W_f are mutually exclusive, thus,

$$P(W) = P(W_f) + P(W_p) = \frac{8}{36} + \frac{134}{495} = \frac{244}{495}$$

C. Expected profit: In this section, I will calculate the expected net profit from a \$90 bet on the Pass Line for a total of n times, where n can be any positive integer. At the outset, I will explain some concepts in probability: expected value, variance, and standard deviation.

Table 2. The Probability of W_i

i	$P(R_i)$	$P(R_i)+P(R_j)$	$P(W_i)$
4	$\frac{3}{36}$	$\frac{3}{36} + \frac{6}{36} = \frac{9}{36}$	$\frac{1}{36}$
5	$\frac{4}{36}$	$\frac{4}{36} + \frac{6}{36} = \frac{10}{36}$	$\frac{2}{45}$
6	$\frac{5}{36}$	$\frac{5}{36} + \frac{6}{36} = \frac{11}{36}$	$\frac{25}{396}$
8	$\frac{5}{36}$	$\frac{5}{36} + \frac{6}{36} = \frac{11}{36}$	$\frac{25}{396}$
9	$\frac{4}{36}$	$\frac{4}{36} + \frac{6}{36} = \frac{10}{36}$	$\frac{2}{45}$
10	$\frac{3}{36}$	$\frac{3}{36} + \frac{6}{36} = \frac{9}{36}$	$\frac{1}{36}$

Expected Value: According to Grinstead and Snell (1997), suppose random variable X can take the value x_1 with the probability of p_1 , the value x_2 with the probability p_2 , and so on, up to the value x_k with the probability p_k ($k=1,2,3,\dots$). Then the expected value of this random variable X is defined as

$$E(X) = x_1p_1 + x_2p_2 + \dots + x_kp_k.$$

Variance: According to Grinstead and Snell (1997), let X be a numerically valued random variable with expected value $E(X)$. Then the variance of X , denoted by $Var(X)$, is.

$$Var(X) = E(X^2) - [E(X)]^2$$

Standard Deviation: According to Grinstead and Snell (1997), the standard deviation of X , denoted by σ_X , is

$$\sigma_X = \sqrt{Var(X)}.$$

We will use the random variable X_i to denote the profit, in dollars, from the i^{th} game. Then X_i takes on either the value 90 when the player wins or the value -90 when the player loses. Therefore,

$$E(X_i) = 90 * \frac{244}{495} + (-90) * (1 - \frac{244}{495}) = -\frac{14}{11}$$

$$E(X_i^2) = 90^2 * \frac{244}{495} + (-90)^2 * (1 - \frac{244}{495}) = 8100$$

$$Var(X_i) = E(X_i^2) - [E(X_i)]^2 = 8100 - (-\frac{14}{11})^2 = \frac{979,904}{121}$$

$$\sigma_{X_i} = \sqrt{Var(X_i)} = \sqrt{\frac{979,904}{121}}$$

Let T_n denote the total profit from playing the game a total of n times. Since X_1, X_2, \dots, X_n are independent and identically distributed random variables with the same mean $-\frac{14}{11}$ and the same standard deviation (square root of $979,904/121$), hence the mean, the variance, and the standard deviation of T_n are as follows:

$$E(T_n) = \sum_{i=1}^n E(X_i) = nE(X_i) = -\frac{14n}{11}$$

$$Var(T_n) = \sum_{i=1}^n Var(X_i) = nVar(X_i) = \frac{979,904n}{121}$$

$$\sigma_{T_n} = \sqrt{Var(T_n)} = \sqrt{\frac{979,904n}{121}}$$

By the Central Limit Theorem, T_n is approximately normally distributed with a mean of $-\frac{14n}{11}$, and a standard deviation of

$$\sqrt{\frac{979,904n}{121}}$$

when n is sufficiently large (at least 30).

Therefore, the chance of making a profit after playing the game a total of n times is

$$P(T_n > 0) = P\left[Z > \left(\frac{0 - \left(\frac{-14n}{11}\right)}{\sqrt{\frac{979,904n}{121}}}\right)\right] \approx P(Z < -0.014\sqrt{n})$$

Using the standard normal table, we can find the value of $P(T_n > 0)$ for all values of n . Table 3 shows the value of $P(T_n > 0)$ for some selected values of n .

Table 3. Probability of Making a Profit

n	$P(T_n > 0)$
100	0.443
400	0.3897
900	0.3372
2,500	0.2420
6,400	0.1292
10,000	0.0808
40,000	0.0023
100,000	0*

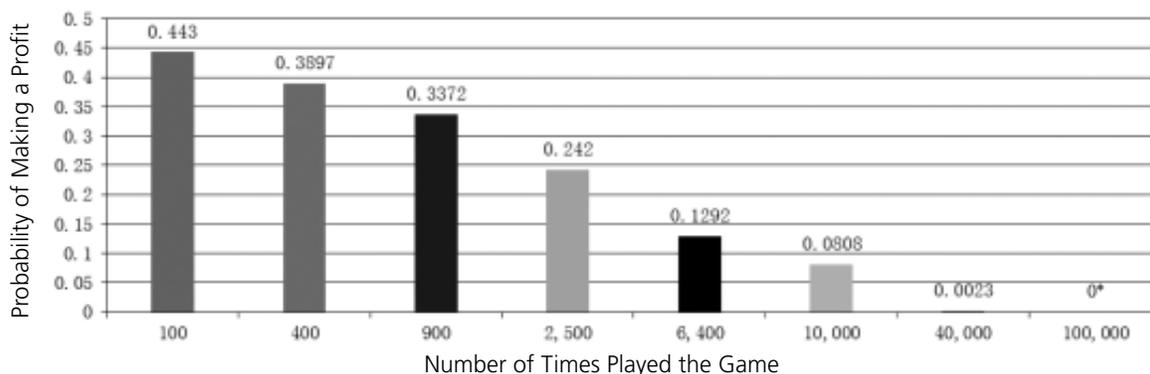
* it is less than .005 of one percent

The results in Table 3 are shown in the following figure, Figure 1.

Conclusion

As shown in Figure 1, if a player plays the game long enough, then the chance for the player to come out ahead (making a profit, that is) is slim to none. The chance of making a living from playing the game regularly is even smaller. If each game takes an average of 1.5 minutes to play, it will take a total of 1,000 hours if a player plays the game a total of 40,000 times. Assuming the player spends an average of 20 hours a week playing this game, he or she would have played the game more than 40,000 times in one year. Therefore, if a player plays the game regularly, he or she will have a less than one fourth of one percent chance to be ahead after one year of playing the game. On the other hand, the casino operators should never worry about losing big money to certain lucky patrons. Considering the size of the sample (pooling all the thousands and millions of patrons together), the chance of meeting a profit goal for the casino is almost certain with a negligible margin of error. In summary, almost no one is going to make money playing craps over a long period of time. The results we have proved in this article are testimony to an old Chinese saying that “if you gamble long enough, you are destined to lose.”

Figure 1. Probability of Making a Profit on a Pass Line Bet



* it is less than .005 of one percent

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